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Numerical Solutions for the Nonlinear Penetration of Magnetic Flux into Type II Superconductors

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The penetration of magnetic flux into a type II superconductor is determined by two coupled, nonlinear, diffusion equations for the flux and temperature. These equations have been written in standard finite difference forms, and in this work solutions have been obtained for values of the time less than a critical value. The latter has been identified with the onset of a physical instability known as a flux jump.

1. INTRODUCTION

The problem considered here concerns the effects of magnetic fields upon type II superconductors, and it arises basically from the practical interest in producing superconducting magnets [1]. One of the major difficulties with these materials is that for specimens above a certain size an increasing magnetic field may suddenly penetrate catastrophically, thereby destroying the superconducting state. This unstable penetration of the field is referred to as a flux jump, and its occurrence is determined by the thermal distribution within the sample. The purpose of the present work was to simulate the conditions inside an infinite slab of material using a computer with a view to predicting the onset of a jump.

The penetration of the magnetic flux density B is determined by a nonlinear diffusion equation. If the sample is subjected to a sweeping external magnetic field the flux diffuses in and produces a local heating of the specimen. The resulting temperature distribution T also satisfies a nonlinear diffusion equation, and in fact the equation for B and T are coupled by the field and temperature dependence of the physical parameters of the material. For an infinite slab, the problem reduces to the one dimension (x, e.g.) perpendicular to the slab faces, and the appropriate equations for B and T are given in the following section.

In solving these equations numerically, the major concern was to ensure that the solutions of the finite difference equations were stable. While this is always important, it was particularly so here because the aim was to predict rapid variations in the 'real' solutions, i.e., flux jumps. Since no general criteria exist for the stability of nonlinear finite difference equations, methods known to be stable for *linear* equations have been used, e.g., iteration of the Crank–Nicolson form of finite difference equation for the temperature distribution.

2. The Basic Equations

The penetration of magnetic flux into a superconductor is described by the equation [2–4]

$$\frac{1}{\mu_0} B \frac{\partial B}{\partial x} = F_p(T) + \eta(T) \int_0^x \frac{\partial B}{\partial t} \, dx. \tag{1}$$

in which μ_0 is a constant, and x is measured from the center of the slab. The functional forms of F_p and η are known from experiment and in the calculations were assumed to be given by the approximate formulas [3, 4]

$$F_p(T) = F_0\{1 - (T/T_c)^2\}^2 \tag{2}$$

and

$$\eta(T) = \eta_0\{1 - (T/T_c)^2\}, \tag{3}$$

where F_0, η_0, and T_c are physical constants. The power $P(x, t)$ generated by the magnetic field is given by

$$P(x, t) = \frac{1}{\mu_0} \frac{\partial B}{\partial x} \int_0^x \frac{\partial B}{\partial t} \, dx, \tag{4}$$

and the temperature distribution produced by the heating is determined by the equation

$$\frac{\partial T}{\partial t} = \frac{K(T)}{S(T)} \frac{\partial^2 T}{\partial x^2} + \frac{1}{S} \frac{\partial K}{\partial x} \frac{\partial T}{\partial x} + \frac{P(x, t)}{S}. \tag{5}$$

The thermal conductivity K and specific heat S were assumed, again from analysis of experimental data [5, 6] of extreme type II materials, to depend upon temperature approximately as

$$K(T) = K_0 T^3 \tag{6}$$

and

$$S(T) = S_0 T^3 \tag{7}$$

where K_0 and S_0 are constants.

At the center of the slab it is required by symmetry that

$$\partial T/\partial x = 0 \qquad \text{and} \qquad B(-x) = B(x). \tag{8}$$

At the surface of the specimen, $x = x_s$ say, the magnetic flux density was increased (by external means) at a steady rate $\dot{B}_{\text{in}}$ from some fixed initial value B_{in}. Hence at time t

$$B(x, t) = B_{\text{in}} + \dot{B}_{\text{in}} t. \tag{9}$$

The boundary condition for the temperature at the surface is determined by the appropriate law for the rate of loss of heat. The latter is usually taken to be proportional to some power of the temperature difference between the surface and the surrounding external temperature T_0, say. For the problem in hand the surrounding fluid is liquid helium, and experimental measurements [7, 8] suggest a cubic law is appropriate for small heat flow, thus

$$[K(\partial T/\partial x)]_{x_s,t} = -Q_0\{T(x_{s,t}) - T_0\}^3 \tag{10}$$

where Q_0 is a constant.

The starting solutions employed were as follows. A small value of the external flux B_{in} equal to 0.1 T was assumed which was well below the flux at which instabilities were expected. The flux distribution inside the specimen was then obtained by direct integration of Eq. (1) with $\partial B/\partial T = 0$, and was

$$B(x, 0) = [B_{\text{in}}^2 - 2\mu_0 F_p(T)(x_s - x)]^{1/2}. \tag{11}$$

Thus flux penetrated into the specimen a distance $(x_s - x')$, where x' satisfied $B(x', 0) = 0$, and for $x < x'$ it was supposed that $B(x, 0) = 0$, so that one has essentially an initial value problem.

The temperature distribution at $t = 0$ was assumed to be uniform and slightly above the temperature of the surroundings. This value of the temperature was used in Eq. (11).

3. Finite Difference Equations

At any particular time, specified in terms of an increment Δt by an integer j ($t = j\,\Delta t$), suppose that the temperature and field distributions $T_j(x)$ and $B_j(x)$ are known. To advance the solutions in time the procedure adopted was to determine firstly $B_{j+1}(x)$, then the power $P_{j+1}(x)$, and finally the new temperature distribution $T_{j+1}(x)$.

It was found that the solutions of the magnetic flux Eq. (1) were much less sensitive than those for the temperature equation to the physical and numerical parameters involved, and a simple forward difference expression for it was found to be satisfactory. Explicitly (1) was written as

$$B_{j+1}(\partial B_{j+1}/\partial x) = f_{j+1}(x) \tag{12}$$

where $f_{j+1}(x)$ is readily identified by comparison with the original. A first approximate solution, $B_{j+1}^{(0)}(x)$ say, was obtained by letting $\partial B/\partial t = 0$, and the result is given by (11) with B_{in} replaced by $\{B_{\text{in}} + \dot{B}_{\text{in}}(j+1)\,\Delta t\}$. A better solution was obtained by evaluating

$$f_{j+1}^{(1)}(x = i\Delta x) = \mu_0 F_p(T_{i,j}) + \mu_0 \eta(T_{i,j})(1/\Delta t)[\phi_{i,j+1}^{(0)} - \phi_{i,j}], \tag{13}$$

where integer i labels the mesh in x, and

$$\phi_{i,j} = \sum_{m=0}^{i} B_{m,j}\,\Delta x. \tag{14}$$

Expression (13) was then repeatedly substituted into the following simple finite difference equation for $B_{j+1}(x)$, and a solution obtained by iteration, where for the l-th iterate

$$B_{m-1,j+1}^{(l)} = B_{m,j+1}^{(l)} - (\Delta x / B_{m,j+1}^{(l)})\, f_{m,j+1}^{(l)}\,. \tag{15}$$

The solution was advanced into the specimen from the surface $x_s = n\,\Delta x$ as far as necessary for B to become zero.

The power distribution generated by the change in B in time Δt is given by Eq. (4), which was written as

$$P_{i,j+1} = (1/2\mu_0)(1/\Delta x\,\Delta t)(B_{i+1,j+1} - B_{i-1,j+1})(\phi_{i,j+1} - \phi_{i,j}). \tag{16}$$

It is clear that the power is only finite in regions where B is nonzero. In fact dP/dx becomes large at the nose of the magnetic field distribution where $B \to 0$, and problems were encountered because of this. These will be discussed in the following section.

The temperature diffusivity Eq. (5) was written in the Crank–Nicolson form of finite difference equation [9]

$$\begin{aligned}
&\frac{1}{\Delta t}\{T_{i,j+1} - T_{i,j}\} \\
&\quad = \frac{K_0}{S_0}\frac{1}{2(\Delta x)^2}\{T_{i-1,j+1} - 2T_{i,j+1} + T_{i+1,j+1} + T_{i-1,j} - 2T_{i,j} + T_{i+1,j}\} \\
&\quad\quad + \frac{1}{8}\frac{1}{(\Delta x)^2}\frac{1}{S_{i,j}}\{K_{i+1,j} - K_{i-1,j}\} \\
&\quad\quad \times \{T_{i+1,j+1} - T_{i-1,j+1} + T_{i+1,j} - T_{i-1,j}\} + \frac{P_{i,j+1}}{S_{i,j}}\,.
\end{aligned} \tag{17}$$

Collecting together terms and rearranging, it can be shown that for the k-th iteration within a Gauss–Seidel scheme [10]

$$T_{i,j+1}^{(k)} = (1/(1+2r))\{(r - r'_{i,j})\,T_{i-1,j+1}^{(k)} + (r + r'_{i,j})\,T_{i+1,j+1}^{(k-1)} + b_{i,j}\} \tag{18}$$

where

$$r = (1/2)(K_0/S_0)(\Delta t/(\Delta x)^2), \tag{19}$$

$$r'_{i,j} = (1/8S_{i,j})(\Delta t/(\Delta x)^2)(K_{i+1,j} - K_{i-1,j}), \tag{20}$$

and

$$b_{i,j} = T_{i,j} + r[T_{i-1,j} - 2T_{i,j} + T_{i+1,j}] + (P_{i,j}/S_{i,j})\,\Delta t + r'_{i,j}(T_{i+1,j} - T_{i-1,j}). \tag{21}$$

Equation (18) is trivially converted to one which is over relaxed, but, as will be discussed later, it was found that for the choice of parameters employed, the optimum relaxation factor was unity.

At the center of the specimen ($i = 0$) the boundary condition $\partial T/\partial x$ implies that $T_{-1,j} = T_{1,j}$ and hence that $\partial K/\partial x = 0$. Thus from (18) and (21)

$$T^{(k)}_{0,j+1} = (1/(1 + 2r))\{2rT^{(k-1)}_{1,j+1} + b_{0,j}\} \tag{22}$$

where

$$b_{0,j} = 2r(T_{1,j} - T_{0,j}) + (\Delta t/S_{0,j})\,P_{0,j} + T_{0,j}\,. \tag{23}$$

At the surface of the specimen ($i = n$) the following finite difference form of Eq. (10) yields

$$T^{(k)}_{n,j+1} = T^{(k)}_{n-1,j+1} - (\Delta x\; Q_0/K_{n,j})\{T^{(k-1)}_{n,j+1} - T_0\}^3. \tag{24}$$

4. Practical Considerations

The appropriate values of the constants occurring in (1)–(10) were known from experimental data, and those used in the present calculations are listed in Table I.

The choice of increment Δx was determined by the fact that near the nose of the magnetic field distribution (see Fig. 1), the derivative $\partial P/\partial x$ became very large. As a result, for small values of n (e.g., $n = 20$), significant oscillations in the temperature distribution were obtained, and it was found necessary to employ n as large as 100. To further aleviate this particular problem the power distribution was rounded at the nose by putting a small value for the power on the x-mesh point preceding it. With $n = 100$ the increment Δx for the specimen thickness employed was 4.16×10^{-6}m, and this was used in all stages of the computation.

The minimum size for Δt was restricted by the computing time available, and a value of $\Delta t = 0.001$ sec was the smallest practicable. This was so much larger than Δx that the conditions for obtaining solutions of the temperature diffusivity equation appeared most unfavorable. Assuming by analogy with the theory of linear equations that an important parameter is r defined by Eq. (19), then with $K_0 = 5 \times 10^{-4}$ and $S_0 = 60$ this has the value

$$r = (1/2)(K_0/S_0)(\Delta t/(\Delta x)^2) \simeq 240. \tag{25}$$

TABLE I

Parameters Employed

Slab thickness	$8.32 \times 10^{-4}\ m$
T_c	$10\ K$
F_0	$0.8 \times 10^{9}\ N m^{-3}$
η_0	$10^{5}\ N m^{-4}\ s$ Tesla^{-1}
K_0	$0.5\ m\ W m^{-1}\ deg^{-4}$
S_0	$60\ J m^{-3}$ deg^{-4}
Q_0	$6 \times 10^{4}\ W m^{-2}$ deg^{-3}
T_0	$4.2\ K$
B_{in}	$0.1\ T$
$\dot{B}_{in}$	$0.7\ T$ sec^{-1}

It is well known in the theory of linear equations that the equivalent quantity $\Delta t/(\Delta x)^2$ must be less than 0.5 for explicit methods, but no such restriction applies to the implicit Crank–Nicolson method (see for example Ref. [11]) and the same was assumed regarding r for nonlinear equations.

Again by analogy with the linear case it is possible to make an estimate of the optimum relaxation factor (ω) to be made in a method of successive over relaxation [10, 12]. With the value of r given by Eq. (25) evaluation of

$$\omega = 1/(1 + (1 - \theta^2)^{1/2}) \qquad \text{with} \qquad \theta = (r/(1 + r)) \cos(\pi/n) \tag{26}$$

yields $\omega = 1.000$, implying that the finite difference equations reduce to the Gauss–Seidel form described above.

Iterative methods were used in order to take advantage of the large number of zeros in the appropriate matrices, especially for the magnetic field diffusion equation. The number of iterations required for a solution of the latter to an accuracy of $10^{-3}\,\%$ was always found to be less than 30, but the convergence of the solutions of the temperature equation was generally poor. In this case the criteria for convergence were first that successive temperatures at several selected values of i should differ by less than $10^{-3}\,\%$, and second, if this was fulfilled, iteration was only terminated provided the sum

$$\frac{1}{n} \sum_{i=1}^{n} | T_{j,i+1}^{(k)} - T_{i,j+1}^{(k-1)} |$$

was less than $10^{-3}\,\%$. A minimum of 80 iterations were made, and computing was stopped if the number exceeded 700.

The equations were coded in FORTRAN for an ICL-KDF9 computer.

5. Results and Discussion

Figure 1 shows some typical results obtained for the magnetic flux distributions through the specimen, and Figs. 2 and 3 illustrate the differences in the temperature distributions obtained for values of $\Delta t = 0.002$, and 0.001 sec, respectively. In view of the computing time restrictions which made it impractible to reduce Δt further, the results obtained for $\Delta t = 0.001$ have been assumed to be reasonable representations of the physical situation. That being so it is clear from Fig. 3 that the rate of temperature rise has increased after time $t = 0.700$ sec, and this is illustrated explicitly in Fig. 4 where the temperature at the point $i = 60$ is plotted as a function of time. Further, after time $t = 0.726$ sec the rise is so rapid that the convergence of the solutions of the temperature equation can not be obtained within the maximum number of iterations specified. These facts are taken as evidence that the rate of temperature increase is running away, thereby allowing rapid entry of the magnetic field through the functional dependence of F_p and η, and hence

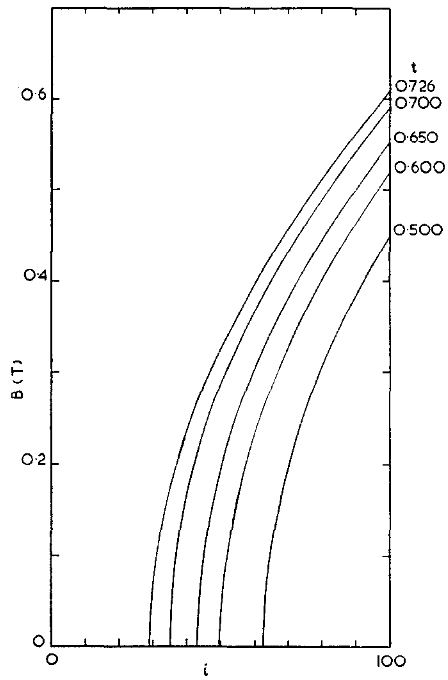

Fig. 1. Solutions for the magnetic flux density distribution in one half of the slab obtained using $\Delta t = 0.001$.

indicating that a flux jump has been initiated at a time approximately equal to $t = 0.726$. The value of the external field when this occurs is obtained from Eq. (9) to be $0.608T$, which compares favorably with the appropriate experimental values obtained by Wipf and Lubell [13].

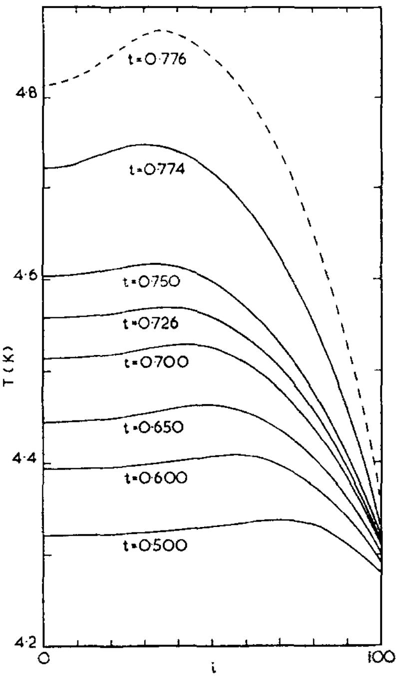

FIG. 2. Temperature distributions in one half of the slab calculated using $\Delta t = 0.002$.

In order to observe the behavior of the magnetic field and temperature distributions during a flux jump it will be necessary to reduce Δt well below 0.001 sec. It is known from experiment that the jump is completed in a time of order 0.001 sec, and hence the inability of the present calculation to obtain convergent iteration for $T(x)$ once the instability has begun is not surprising. It is intended to employ smaller Δt values when a more powerful computer becomes available to the authors in the near future. It may then be possible to follow the rapid temperature rise until the temperature reaches the constant T_c of Eqs. (2) and (3), at which point the behavior is no longer described by the equations given in Section 2.

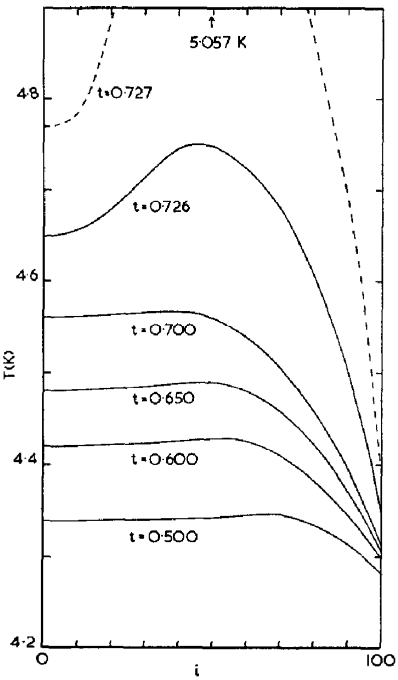

FIG. 3. Temperature distributions calculated with $\Delta t = 0.001$.

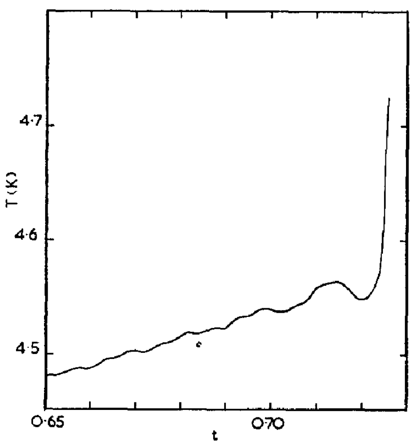

FIG. 4. The temperature at the position x specified by $i = 60$ as a function of time.

6. Conclusions

The main difficulties of the problem considered here arose from the nonlinearity of the equations and the consequent restrictions on computing time imposed by this. In order to avoid further complications as much as possible the numerical methods employed were simple and conventional, although it was found necessary to use some insight into the physics, for example in slightly modifying the power distribution. The onset of the instability inferred from the results obtained for $\Delta t = 0.001$ is in good agreement with experiment, and it is concluded that meaningful solutions can be obtained with the methods used even when Δt is so large.

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